Linear Polynomial Estimator: The State Observer

Dr. David Di Ruscio
Telemark university college
Department of process automation
N-3914 Porsgrunn, Norway
Fax: +47 35 57 52 50
Tel: +47 35 57 51 68

November 3, 2009

Abstract

A simple view of the state observer and the equivalence with the problem of specifying the observer polynomial, e.g., as Butterworth filter polynomials is presented.

Key words: State observer; Observer canonical form; Butterworth filters; Linear systems.

1 Introduction

The problem of state estimation of linear dynamic systems is addressed in this paper. Often linear quadratic optimal state estimation is used. It may be shown that there exists weighting matrices in the linear quadratic performance objective that corresponds to a set of prescribed eigenvalues of the observer, se. e.g., Di Ruscio and Balchen (1990).

The classical Luenberger type of observer where the observer gain matrix is chosen directly so that the eigenvalues are prescribed may be a simple choice in some circumstances. We will in this paper concentrate on state estimation by using Luenberger type observers. See Kailath (1980) for a view on observers.

It a classic result that the observer gain matrix, K, may be chosen so that the eigenvalues of the observer may be located arbitrarily in the left half part of the complex plane (continuous systems). But how to locate the eigenvalues is an intricate question in many circumstances. This problem is addressed.

2 State estimation and the observer

2.1 State observer for continuous time systems

A problem description is presented in the following.

Given a system described by the linear or linearized state space model

$$\dot{x} = Ax + Bu + v, \quad (1)$$
$$y = Cx + Du + w. \quad (2)$$

A state observer for the system in (1) and (2) have the following structure

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \tag{3}$$

$$\hat{y} = C\hat{x} + Du. \tag{4}$$

The response of the observer (filter) from the inputs y and u to the output $\hat{y}$ is described by

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky, \tag{5}$$

$$\hat{y} = C\hat{x} + Du. \tag{6}$$

The system matrix of the observer, i.e. $A - KC$, is of central importance in the observer. The eigenvalues, $\lambda_i \ \forall \ i = 1, \ldots, n$, of this matrix may be placed arbitrarily by sufficiently choosing the observer gain matrix, K, (modal observability assumed). The eigenvalues should be placed in the left part of the complex plane in order for the observer to be stable and to ensure that the estimate $\hat{x}(t)$ converges to the true state $x(t)$ in a mean sense as time approach infinity, i.e. $t \to \infty$. The problem of finding the gain matrix, K, such that the eigenvalues of the matrix $A - KC$ is prescribed is equivalent to finding the observer gain matrix K so that the coefficients in the characteristic equation $|\lambda I_n - (A - KC)|$ is prescribed.

Consider the special case in which all poles are specified to be $-\lambda_i \ \forall \ i = 1, 2, \ldots, n$, i.e.,

$$|\lambda I_n - (A - KC)| = \Pi_{i=1}^{n}(\lambda + \lambda_i) = \lambda^n + c_1\lambda^{n-1} + \cdots c_{n-1}\lambda + c_n. \tag{7}$$

The problem addressed in this paper is to find the observer gain matrix K so that the characteristic polynomial in (7) is prescribed, i.e. with fixed polynomial coefficients

$$c_i \ \forall \ i = 1, \ldots, n. \tag{8}$$

without eigenvalue computations.

2.2 State observer for discrete time systems

Given a system described by the linear or linearized state space model

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{9}$$

$$y_k = Cx_k + Du_k + w_k. \tag{10}$$

A state observer for the system in (9) and (10) have the following structure

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + K(y_k - \bar{y}_k) \tag{11}$$

$$\bar{y}_k = C\bar{x}_k + Du_k. \tag{12}$$

The results for finding the modal observer gain presented in this paper for continuous time systems may directly be used for discrete time systems. The only difference is to locate the eigenvalues of the system matrix, A, in the discrete system inside the unit circle. The corresponding coefficients in the characteristic polynomial $|\lambda I_n - (A - KC)|$ should be specified instead of the continuous versions of the coefficients. Discrete time systems is therefore not discussed further in this paper.

3 Observer canonical form

3.1 How to transform a model to observer canonical form

Assume given a state space model

$$\dot{x} = Ax + Bu, \tag{13}$$
$$y = Cx + Du. \tag{14}$$

This model may be transformed to observer canonical form by the state transformation

$$x = T_o x_o, \tag{15}$$

where x_o is the state in the observer form canonical state space model, and the transformation matrix T is given by

$$T_o = (M^T O_n)^{-1} = O_n^{-1}(M^T)^{-1} \tag{16}$$

where O_n is the observability matrix of the pair (A, C) and $M \in \mathbb{R}^{n \times n}$, is an upper triangular Toepeliz matrix with $\begin{bmatrix} 1 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}$ in the first row. Hence, M is defined in terms of the coefficients, $a_i \; \forall \; i = 1, \ldots, n$, of the characteristic polynomial of the system matrix, A, i.e.,

$$|\lambda I_n - A| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n. \tag{17}$$

The matrix M in the transformation is defined as

$$M = \begin{bmatrix} 1 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 1 & \ddots & & a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{18}$$

It should be noticed that the toeplitz matrix M^T is related to the inverse of the observability matrix of canonical observer form matrices, A_o and C_o. This relationship and more details of the matrix M is indicated later in this section.

The transformed model

$$\dot{x}_o = A_o x_o + B_o u, \tag{19}$$
$$y = C_o x_o + D_o u. \tag{20}$$

where

$$A_o = T_o^{-1} A T_o, \;\; B_o = T_o^{-1} B, \;\; C_o = C T_o, \;\; D_o = D, \tag{21}$$

is now on so called observer canonical form.

The observer canonical form of an n-th order observable state space model is given by

$$A_o = \begin{bmatrix} -a_1 & 1 & \cdots & 0 & 0 \\ -a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1} & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{bmatrix}, \;\; C_o = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{22}$$

Matrix B_o is a matrix in general full of parameters.

The algorithm presented in this section for computing the observer canonical form is simply implemented in MATLAB as described in the following m-file function

```
function [Ao,Bo,Co,T]=ss2ocf(A,B,C)
% ss2cof
% [Ao,Bo,Co,T]=ss2ocf(A,B,C)
% Transform a state space model
% dot(x)=Ax + Bu, y=Cx
% with the transformation, x=T xo
% to observer canonical form
% dot(xo)=Ao xo + Bo u, y=Co xo
%
a=poly(A);
n=length(a);
On=obsv(A,C);
M=triu(toeplitz([1,a(2:n-1)]))
T=inv(M'*On);
Ao=inv(T)*A*T;
Bo=inv(T)*B;
Co=C*T;
% END SS2OCF
```

Often the subscript o on A is omitted. Hence, we only use A as a symbol for the system matrix A_o on observer form. The algorithm presented in this section works for single output systems. However, it is possible to extend the algorithm to multi output systems. A method for the transformation of a MIMO system to observable canonical form is implemented in the DSR Toolbox for MATLAB function **ss2cf.m**.

3.2 Some details about observer and controller canonical forms

The relationship between the Toeplitz upper triangular matrix M in Equation (18) and the observability matrix of the canonical observer form system is as follows

$$M^T = (O_n^o)^{-1}, \;\; M^{-T} = O_n^o, \tag{23}$$

where O_n^o is the observability matrix for the pair A_o, C_o.

Furthermore the transformation $T_c = C_n M$ and $x = T_c x_c$ will transform the model $\dot{x} = Ax + Bu$, $y = Cx$ to controller form and where x_c is the state in the controller form model. Here C_n is the controllability matrix for the pair A, B. Define C_n^c for the controllability matrix for the controller form matrices A_c, B_c. Then $M = (C_n^c)^{-1}$.

All this may be proved from the Hankel matrix, H_n, with impulse responses $h_i = CA^{i-1}B$ $\forall$ $i = 1, \ldots, n+1$. From realization theory we have that $H_n = O_n C_n = O_n^o C_o^c = O_n^c C_n^c$. Hence, the realizations (A, B, C), (A_o, B_o, C_o) and (A_c, B_c, C_c), i.e., the original model, the observer canonical form model and the controller canonical form model, have the same impulse responses (Markov parameters).

3.3 How to design the observer gain for a system on observer form

Consider now a state observer constructed from a model on observer form, i.e., the observer,

$$\dot{\hat{x}}_o = (A_o - K_oC_o)\hat{x}_o + (B_o - K_oD_o)u + K_oy, \quad (24)$$
$$\hat{y} = C_o\hat{x}_o + Du. \quad (25)$$

The stability of the observer is determined by the eigenvalues of the matrix $A - KC$ which simply is given by

$$A_o - K_oC_o = \begin{bmatrix} -(a_1+k_1) & 1 & \cdots & 0 & 0 \\ -(a_2+k_2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(a_{n-1}+k_{n-1}) & 0 & \cdots & 0 & 1 \\ -(a_n+k_n) & 0 & \cdots & 0 & 0 \end{bmatrix}, K_o = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-1} \\ k_n \end{bmatrix}. \quad (26)$$

One advantage of the observer canonical form is that the characteristic polynomial of the observer system matrix $A - KC$ simply is given by

$$|\lambda I_n - (A - KC)| = \lambda^n + (k_1 + a_1)\lambda^{n-1} + (k_2 + a_2)\lambda^{n-2} + \cdots + (k_{n-1} + a_{n-1})\lambda + k_n - a_n. \quad (27)$$

One point of the above is that it is in particular simple to find the observer gain matrix K_o (and thereafter K) which results in a specified prescribed characteristic polynomial, $|\lambda I_n - (A - KC)|$, of the observer system matrix $A - KC$.

Assume that a prescribed characteristic polynomial of the observer is specified as

$$|\lambda I_n - (A - KC)| = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n, \quad (28)$$

where $c_i \ \forall\ i = 1, \ldots, n$ are the prescribed coefficients of the observer characteristic polynomial. The observer gain matrix is then simply defined as

$$K_o = \begin{bmatrix} c_1 - a_1 \\ c_2 - a_2 \\ \vdots \\ c_{n-1} - a_{n-1} \\ c_n - a_n \end{bmatrix} = c - a. \quad (29)$$

where c is a vector of the prescribed observer polynomial coefficients as indicated, and a is a vector of the coefficients of the characteristic polynomial, $|\lambda I_n - A|$ also as indicated.

The observer gain K_o is the gain for an observer constructed for a system on observer canonical form as in (5) and (6). As we see the observer gain, K_o, is purely defined in terms of the coefficients of the characteristic polynomial of the system matrix A, i.e., $|\lambda I_n - A|$, and the coefficients of the prescribed characteristic polynomial of the observer, i.e., $|\lambda I_n - (A - KC)| = 0$.

It is interesting that K_o may be found through the use of the Cayley Hamiltin theorem. We have the following Lemma,

Lemma 3.1
The observer gain in the observer canonical form model is given by

$$K_o = CP(A)O_n{-}1, \quad (30)$$
$$K_o = rot90(K_o), \quad (31)$$

where $P(A)$ is the evaluation of A of the characteristic equation of the prescribed characteristic polynomial $|\lambda I_n - (A - KC)$, i.e.,

$$P(A) = A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I_n, \tag{32}$$

where $c_i \ \forall \ i = 1, \ldots, n$ is the prescribed coefficients of the characteristic polynomial.

Proof 3.1
Evaluation of the system matrix A through the prescribed charateristic matrix of the observer is defined as

$$\begin{aligned} P(A) = |\lambda I - (A - KC)| &= A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I_n \\ &= A^n + (k_1 + a_1) A^{n-1} + \cdots + (k_{n-1} + a_{n-1}) A + (k_n + a_n). \end{aligned} \tag{33}$$

Evaluate A in its own characteristic polynomial gives, i.e. using the Cayley Hamilton theorem

$$A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I_n = 0, \tag{34}$$

which gives

$$A^n = -a_1 A^{n-1} - \cdots - a_{n-1} A - a_n I_n. \tag{35}$$

Substitute (35) into (33) gives

$$P(A) = -a_1 A^{n-1} - \cdots - a_{n-1} A - a_n I_n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I_n \tag{36}$$

$$= k_1 A^{n-1} + \cdots + k_{n-1} A + k_n I_n \tag{37}$$

Multiply (37) with the output matrix C gives

$$CP(A) = \begin{bmatrix} k_n & k_{n-1} & \cdots & k_2 & k_1 \end{bmatrix} O_n, \tag{38}$$

and finally

$$\tilde{K}_o = CP(A) O_n^{-1}, \tag{39}$$

Finally we have to rotate the results 90 degrees, ore reshape the result $\tilde{K}$, in order to obtain the correct column vector K_o. It is also possible to multiply with the inverse of the reversed observability matrix in order to get the observer gain correct in the first instance.

4 The observer gain

The observer gain, K_o, for the observer canonical form model may be found as in the above section. If it is necessary to scale the gain matrix K_o in the observer form back to the original coordinate system, i.e., for an observer for the system in (13) and (14), we simply use the transformation

$$K = T_o K_o, \tag{40}$$

where K_o is the observer gain for the canonical observer form model given by Equation (39) and T_o is the transformation matrix given by Equation (16). Hence, K given by (40) is the observer gain for an observer constructed for a model as in (9) and (10), i.e. the observer gain in the observer (24) and (25).

This may with advantage be formulated more compact as follows

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}. \tag{41}$$

Then we have

$$K = T_o(c - a) = O_n^{-1} M^{-T}(c - a), \tag{42}$$

where we have used the transformation T_o as given in equation (16).

It should be noted with interest that this formula for the observer gain matrix, K, is the dual of a well known formula known as the Bass-Gura formula for the state feedback gain matrix, se Kailath (1980) p. 199 eq. (13). Formula (42) is equivalent to eq. (12) in Kailath (1980).

The algorithm presented in this section is implemented in the MATLAB **lpe** function given as

```
function [K,Ao,Co,Ko,T]=lpe(A,C,c)
% LPE Linear Polynomial Estimator
% [K,Ao,Co,Ko,T]=lpe(A,C,c)
% Purpose: Compute the observer gain matrix K which results
% in a prescribed characteristic polynomial of the observer.
% On input
% A, C - matrices in the linear state space model
% c    - an n+1 dimensional vector with the coefficients of char. poly. coefs.
% On output
% K - The observer gain matrix
%
n=size(A,1);
% Observer canonical form
B=C';       % Dummy B
[Ao,Bo,Co,T]=ss2ocf(A,B,C);
a=poly(A);
Ko=c(2:n+1)'-a(2:n+1)';   % Observer gain in observer form
K=T*Ko;  % Scale to observer gain in original coordinate system
% END lpe
```

Note that the eigenvalues and the poles of the observer is given by the roots of the characteristic equation, i.e., $|\lambda I_n - (A - KC)| = 0$. We will in the next chapter show that it with the use of this is in particular simple to find the observer gain matrix K so that the observer have the same characteristic polynomial as the Butterworth filter polynomial, B_n. Note also that the observer gain matrix then is constructed without any eigenvalue computations.

It is worth noticing that the observer gain K is obtained directly as the last column of O_n^{-1} times the $P(A)$ matrix. This is an estimator version of Ackermans formula for feedback gain. We have the following Lemma

Lemma 4.1
Given specified observer polynomial coefficients, c_i, $\forall$, $i = 1, \dots, n$. The corresponding observer gain K is given by,

$$\begin{aligned} P(A) &= A^n + c_1 A^{n-1} + \cdots + c_{n-1}A + c_n I_n, & (43)\\ Oi &= O_n^{-1}, & (44)\\ K &= POi(:, n). & (45) \end{aligned}$$

The above Lemma may also be formulated by defining the last column in, O_n^{-1}, as the vector q_n. Hence,

$$\begin{aligned} q_n &= O_n^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, & (46)\\ K &= Pq_n & (47) \end{aligned}$$

This formula for the observer gain matrix, K, is the dual of Ackerman's formula for computing the modal feedback gain matrix and also presented in Equation (23a) in Kailath (1980). However, the above formulas, Equations (45) and (47) for the linear polynomial estimator gain, K, is novel. In Kailath (1980) the above is presented as an Exercise 4.1.1, p. 267 but without solution.

As we see the formula (47) is independent of the coefficients, a, in the characteristic polynomial $|\lambda I_n.A|$ of the open loop system matrix, A. Only the knowledge of a pair of observable system matrices A and C in additions to the prescribed polynomial coefficients in the vector c is needed in order to find the estimator gain K. However, note that the formula presented in equation (42) is dependent on the coefficients of the characteristic polynomial, a, of the open loop system matrix, A.

This may be implemented in MATLAB as shown in the **lpe2.m** function.

```
function [K,Ko,P]=lpe2(A,C,c)
% LBE Linear Butterworth observer and Estimator
% [K,Ko,P]=lpe2(A,C,c)
[m,n]=size(C);
% Compute the polynomial evaluation, P(A)
P=polyvalm(c,A);
On=obsv(A,C);
% Observer gain in observer form model..
Ko=C*P*inv(On);
Ko=rot90(Ko);    % Gain in observer form (not needed)
Oi=inv(On);
K=P*Oi(:,n );    % Estimator gain in original coordinate system.
```

5 The state observer pole polynomial

We will here suggest a simple choice for the pole polynomial of the observer. The location of the poles and in particular the coefficients, c_i $\forall$ $i = 1, \dots, n$, in this polynomial is essential. We suggest for the reason of simplicity and in order to have nice smooth set-pint responses without overshoot to locate all n poles, i.e., $s = -\frac{1}{T}$ as a function of only one prescribed time constant T.

Assume that a prescribed characteristic polynomial of the observer is specified as

$$C_n = |\lambda I_n - (A - KC)| = (s + \frac{1}{T})^n = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n. \tag{48}$$

For the different orders, $n = 2$, $n = 3$ and $n = 4$ we have

$$C_2 = |\lambda I_2 - (A - KC)| = (s + \frac{1}{T})^2 = s^2 + \frac{2}{T}s + \frac{1}{T^2}, \tag{49}$$

$$C_3 = |\lambda I_3 - (A - KC)| = (s + \frac{1}{T})^3 = s^3 + \frac{3}{T}s^2 + \frac{3}{T^2}s + \frac{1}{T^3}, \tag{50}$$

$$C_4 = |\lambda I_4 - (A - KC)| = (s + \frac{1}{T})^4 = s^4 + \frac{4}{T}s^3 + \frac{6}{T^2}s^2 + \frac{4}{T^3}s + \frac{1}{T^4}. \tag{51}$$

It is well known that working with polynomials and in particular polynomials with multiple roots is an ill conditioned problem due to rounding off errors. In case of numerical problems the above simple choice may be modified so that the time constants (eigenvalues) are spread somewhat around the prescribed time constant T.

6 State observer and the Butterworth filter

We will in this section show that the coefficients, c_i, $\forall$, $i = 1, \ldots, n$ may be chosen equal to the coefficients in an n-th order Butterworth filter polynomial. Those Butterworth filter coefficients are taken from Haugen (2009).

6.1 2nd order system

The observer canonical form of an 2nd order observable state space model is given by

$$A = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{52}$$

The observer gain matrix is

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \tag{53}$$

and the stability of the observer is given by the matrix

$$A - KC = \begin{bmatrix} a_1 - k_1 & 1 \\ a_2 - k_2 & 0 \end{bmatrix}. \tag{54}$$

The characteristic equation of the observer is then given by

$$|\lambda I_2 - (A - KC)| = \begin{bmatrix} \lambda + k_1 - a_1 & -1 \\ k_2 - a_2 & \lambda \end{bmatrix} = \lambda^2 + (k_1 - a_1)\lambda + k_2 - a_2. \tag{55}$$

A 2nd order Butterworth filter have the characteristic polynomial

$$B_2(s) = T^2 s^2 + \sqrt{2}Ts + 1 = T^2(s^2 + \frac{\sqrt{2}}{T}s + \frac{1}{T^2}). \tag{56}$$

Simply comparing the characteristic equation of the state observer, Equation (55) and the 2nd order Butterworth filter polynomial, Equation (56), shows that they are equivalent if

$$k_1 - a_1 = \frac{\sqrt{2}}{T}, \; k_2 - a_2 = \frac{1}{T^2}. \tag{57}$$

This gives the equivalent observer gain without any eigenvalue computations, i.e.,

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a_1 + \frac{\sqrt{2}}{T} \\ a_2 + \frac{1}{T^2} \end{bmatrix}. \tag{58}$$

6.2 3rd order system

The observer canonical form of an 3rd order observable state space model is given by

$$A = \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ a_3 & 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \tag{59}$$

The observer gain matrix is

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}, \tag{60}$$

and the stability of the observer is given by the matrix

$$A - KC = \begin{bmatrix} a_1 - k_1 & 1 & 0 \\ a_2 - k_2 & 0 & 1 \\ a_3 - k_3 & 0 & 0 \end{bmatrix}. \tag{61}$$

The characteristic equation of the observer is then given by

$$|\lambda I_3 - (A - KC)| = \begin{bmatrix} \lambda + k_1 - a_1 & -1 & 0 \\ k_2 - a_2 & \lambda & -1 \\ k_3 - a_3 & 0 & \lambda \end{bmatrix} = \lambda^3 + (k_1 - a_1)\lambda^2 + (k_2 - a_2)\lambda + k_3 - a_3. \tag{62}$$

A 3rd order Butterworth filter have the characteristic polynomial

$$B_3(s) = (Ts+1)(T^2s^2 + Ts + 1) = T^3s^3 + 2T^2s^2 + 2Ts + 1 = T^3(s^3 + \frac{2}{T}s^2 + \frac{2}{T^2}s + \frac{1}{T^3}). \tag{63}$$

Simply comparing the characteristic equation of the state observer, Equation (55) and the 2nd order Butterworth filter polynomial, Equation (56), shows that they are equivalent if

$$k_1 - a_1 = \frac{2}{T}, \; k_2 - a_2 = \frac{2}{T^2}, \; k_3 - a_3 = \frac{1}{T^3}. \tag{64}$$

This gives the equivalent observer gain without any eigenvalue computations, i.e.,

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a_1 + \frac{2}{T} \\ a_2 + \frac{2}{T^2} \\ a_3 + \frac{1}{T^3} \end{bmatrix}. \tag{65}$$

7 Examples

Example 7.1
Given a system described by

$$\dot{x} = ax + bu, \tag{66}$$
$$y = x + y_0, \tag{67}$$

where $a = -2$, $b = 1$ *and* y_0 *is an unknown slowly varying trend parameter.*

The problem addressed in this example is to use a state estimator in order to both estimate the state, x*, and the unknown slowly varying trend,* y_0*.*

The constant or slowly varying parameter may be modeled by

$$\dot{y}_0 = 0. \tag{68}$$

An equivalent augmented model for this system is then

$$\begin{bmatrix} \dot{x} \\ \dot{y}_0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y_0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u. \tag{69}$$

Hence, the augmented model is of the form

$$\dot{x} = Ax + Bu, \tag{70}$$
$$y = Cx, \tag{71}$$

where

$$x := \begin{bmatrix} x \\ y_0 \end{bmatrix}, \; A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} b \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{72}$$

The time constant in the system model is given by $T_c = -\frac{1}{a} = 0.5$.

We want the observer to be faster than the open loop system, say 4 times faster than the open loop system. Hence, we specify the observer time constant to be approximately $T = T_c/4 = 0.125$.

The time constants of the observer is given by the eigenvalues of the system matrix, $A - KC$*, in the observer,*

$$A - KC = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} a - k_1 & -k_1 \\ -k_2 & -k_2 \end{bmatrix}. \tag{73}$$

The characteristic polynomial is then given by

$$|\lambda I_2 - (A - KC)| = | \begin{bmatrix} \lambda + k_1 - a & k_1 \\ k_2 & \lambda + k_2 \end{bmatrix} | = \lambda^2 + (k_1 + k_2 - a)\lambda - k_2 a = 0. \tag{74}$$

Hence, we have that the coefficients c_1 *and* c_2 *in the prescribed estimator polynomial coefficient vector,* $c = \begin{bmatrix} 1 & c_1 & c_2 \end{bmatrix}$ *is given by*

$$c_1 = k_1 + k_2 - a, \; c_2 = -ak_2. \tag{75}$$

This gives the observer gain in terms of the prescribed observer polynomial coefficients as follows

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} c_1 + a - k_2 \\ -\frac{c_2}{a} \end{bmatrix}. \tag{76}$$

If a 2nd order Butterworth polynomial, $B_2 = T^2(s^2 + \frac{\sqrt{2}}{T}s + \frac{1}{T^2})$*, is chosen then we obtain the observer gain by specifying* $c_1 = \frac{\sqrt{2}}{T}$ *and* $c_2 = \frac{1}{T^2}$*. Hence,*

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{T} + a + \frac{1}{aT^2} \\ -\frac{1}{aT^2} \end{bmatrix} = \begin{bmatrix} -1.1716 \\ 2 \end{bmatrix}. \tag{77}$$

In order not to have high overshot in the observer responses, as is a disadvantage by using Butterworth polynomials, we simply specify the characteristic polynomial to have real double roots, $C_2 = (\lambda^2 + \frac{1}{T})^2 = \lambda^2 + \frac{2}{T}\lambda + \frac{1}{T^2}$*. Hence we specify the observer polynomial coefficients* $c_1 = \frac{2}{T}$ *and* $c_2 = \frac{1}{T^2}$*. Hence,*

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{T} + a + \frac{1}{aT^2} \\ -\frac{1}{aT^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{78}$$

It is worth to notice that $k_1 = 0$ *in this last choice which implies that the estimate* $\hat{x}_1$ *is ballistic and not directly updated from* y*.*

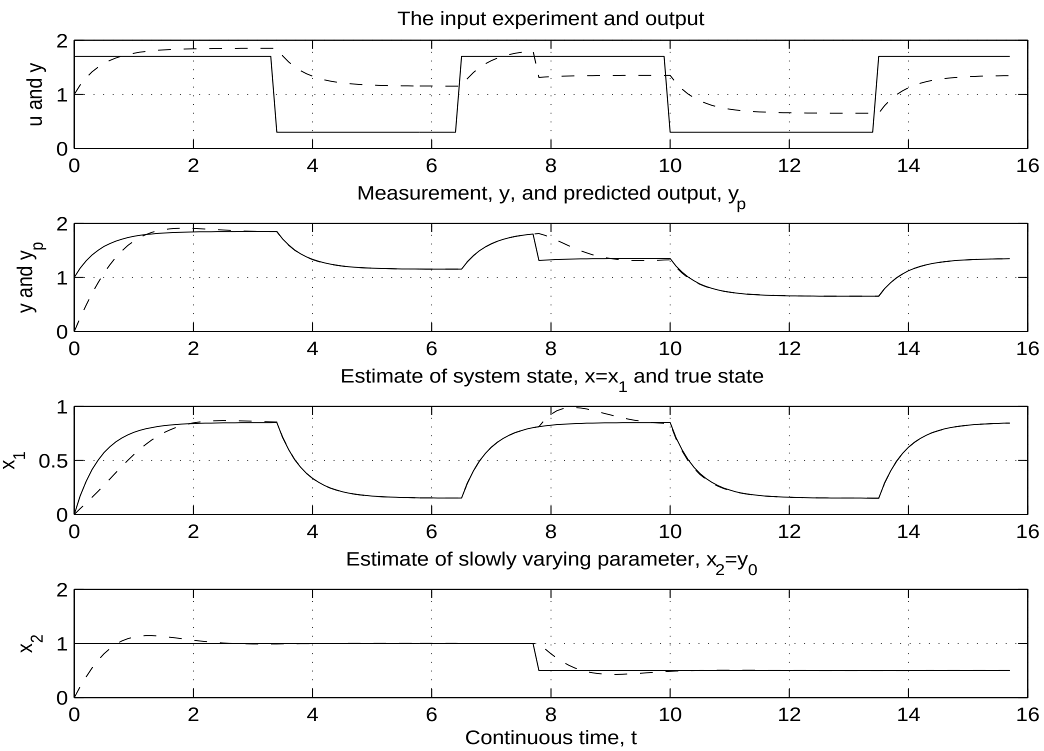

Figure 1: Simulation of the system in Example 7.1 with the state estimator in parallel. Butterworth observer polynomial with gain as in Equation (77).

Tasks solved in this example

1. *Specify that the time constants in the observer should be 4 times faster than the open loop system.*

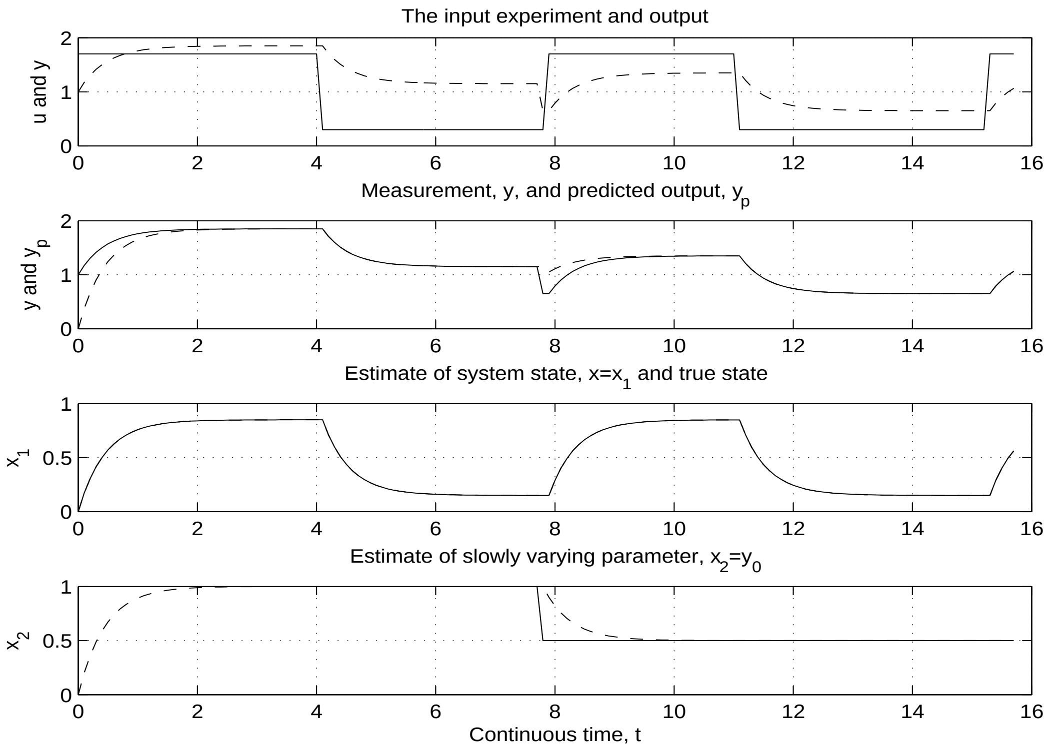

Figure 2: Simulation of the system in Example 7.1 with the state estimator in parallel. Observer polynomial with double time constants and gain as in Equation (78).

2. *Find the state estimator polynomial coefficients so that the polynomial is similar to a 2nd order Butterworth polynomial. Find the corresponding observer gain, K.*

3. *Find the state estimator polynomial coefficients so that the polynomial is similar to a 2nd order polynomial with double eigenvalues in $\lambda_1 = \frac{1}{T}$ $\lambda_2 = \frac{1}{T}$. Find the corresponding observer gain, K.*

4. *Simulate the system with the state observer in parallel.*

8 Concluding remarks

It is shown that the observer gain matrix simply may be constructed from prescribed observer polynomial coefficients.

References

- Di Ruscio, D and J.G. Balchen (1990). A Schur method for designing LQ-optimal systems with prescribed eigenvalues. Modeling, Identification and Control, vol. 11, pp. 55-72.

- Kailath, T. (1980). Linear Systems. Prentice Hall.

- Haugen, F. (2009). Lecture notes in Models, Estimation and Control. TechTeach.