Final Exam (70%) SCE3006 Advanced Control with Implementation Monday December 16th, 2013 kl. 9.00 - 12.00 SKIP THIS FRONT PAGE AND REPLACE WITH STANDARD FROM WORD FILE

David Di Ruscio

December 12, 2013

Task 1 (20%) (Continuous time optimal control)

a) Consider a system described by a discrete state space model

$$\dot{x} = Ax + Bu, \tag{1}$$

$$y = Dx. (2)$$

Assume an infinite horizon LQ control objective as follows

$$J = \frac{1}{2} \int_0^\infty [x^T Q x + u^T P u] dt, \qquad (3)$$

where $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are symmetric weighting matrices.

Find the optimal control which minimize the objective, J, given by (3) subject to the model (1) and (2).

The solution should consist of:

- 1. An expression for the optimal control vector, u^* .
- 2. A Riccati equation.
- b) Consider now a scalar SISO system described by the model

$$\dot{x} = ax + bu, \tag{4}$$

$$y = x, (5)$$

where a and b are scalar model parameters.

Assume an infinite horizon LQ control objective as follows

$$J = \frac{1}{2} \int_0^\infty [qx^2 + pu^2] dt,$$
 (6)

where q and p are scalar weighting parameters.

Find the optimal control which minimize the objective, J, given by (6) subject to the model (4) and (5).

The solution should consist of:

- 1. An expression for the optimal control vector, u^* .
- 2. A Riccati equation.
- c) The LQ optimal control problem in step 1b) above can be solved analytically. Discuss and perform the following:
 - 1. Find an analytical expression for the solution of the Riccati equation in step 1b) above.

- 2. Find an analytical expression for the feedback gain g in the optimal control vector, $u^* = gx$.
- d) The closed loop system in step 1b) and 1c) above may be expressed by $\dot{x} = a_{cl}x$ where a_{cl} is the eigenvalue of the closed loop system.
 - 1. Discuss the relationship between the closed loop eigenvalue a_{cl} as a function of the weighting parameters q and p?
 - 2. Discuss the stability properties of the closed loop system in terms of the weighting parameters q and p?

Task 2 (10%) (Discrete time optimal control)

Given an LQ optimal criterion defined over the time interval $0 \le k \le \infty$, i.e.,

$$J_0 = \frac{1}{2} \sum_{k=0}^{\infty} [x_k^T Q x_k + u_k^T P u_k],$$
(7)

where $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are symmetric weighting matrices.

a) Assume that the system is modeled with a state space model, i.e.,

$$x_{k+1} = Ax_k + Bu_k, (8)$$

Find the optimal controller which are minimizing the objective J_0 given by (7) and subject to the model (8).

The solution should consist of:

- 1. An expression for the optimal control u_k^* .
- 2. A discrete Riccati equation.
- **b**) Assume that the system is given by the scalar system

$$x_{k+1} = ax_k + bu_k, (9)$$

and given the objective

$$J_0 = \frac{1}{2} \sum_{k=0}^{\infty} [qx_k^2 + pu_k^2].$$
 (10)

Find an analytic solution to the LQ optimal control problem. Hint: You may use the results from Task 2a) above.

Task 3 (30%) (Discrete time LQ optimal control with Integral Action)

We are in this task to study an LQ optimal controller for a system described by the state space model

$$x_{k+1} = Ax_k + Bu_k + v, \tag{11}$$

$$y_k = Dx_k + w, \tag{12}$$

where v and w are constant and unknown disturbances.

Subject to the above state space model we want to design an LQ optimal controller which minimizes the following LQ criterion

$$J_{i} = \frac{1}{2} \sum_{k=i}^{\infty} ((r - y_{k})^{T} Q(r - y_{k}) + \Delta u_{k}^{T} P \Delta u_{k}).$$
(13)

where $\Delta u_k = u_k - u_{k-1}$ and r is a constant reference vector. Q and P are symmetric and positive semidefinite matrices.

a) Show that it is possible to write the model in (11) and (12) on deviation form, i.e.,

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \tag{14}$$

$$\Delta y_k = D\Delta x_k, \tag{15}$$

where

$$\Delta x_k = x_k - x_{k-1}, \ \Delta u_k = u_k - u_{k-1}, \ \Delta y_k = y_k - y_{k-1}.$$
(16)

What can be gained by doing this?

b) Show that the model in (14) and (15) can be written as follows

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \tag{17}$$

$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \tag{18}$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ r - y_{k-1} \end{bmatrix}, \quad \tilde{y}_k = r - y_k.$$
(19)

Here you should define the matrices \tilde{A} , \tilde{B} and \tilde{D} .

c) The state space model in 2b) and the LQ criterion in (13) defines a standard discrete LQ optimal control problem of the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k, \qquad (20)$$

$$\tilde{y}_k = \tilde{D}\tilde{x}_k, \tag{21}$$

with LQ criterion

$$J_i = \frac{1}{2} \sum_{k=i}^{\infty} (\tilde{y}_k^T Q \tilde{y}_k + \tilde{u}_k^T P \tilde{u}_k), \qquad (22)$$

where we for simplicity has defined

$$\tilde{u}_k = \Delta u_k. \tag{23}$$

Write down the solution to the LQ optimal control problem, of the form

$$\tilde{u}_k^* = \tilde{G}\tilde{x}_k. \tag{24}$$

the solution should consist of:

- 1. A discrete Riccati equation
- 2. an expression for the controller matrix \tilde{G} .
- 3. Write down an expression for the actual control of the form

$$u_k = f(\cdot) \tag{25}$$

which are to be used in order to control the process.

Task 4 (10%) (Receding horizon optimal control)

Given a system described by the continuous linear state space model

$$\dot{x} = Ax + Bu. \tag{26}$$

We want to study the LQ optimal control problem as described in the following. Consider the LQ optimal control objective defined over a receding horizon from present time t to a future time instant t + T, i.e.,

$$J = \frac{1}{2}x(t+T)^{T}Sx(t+T) + \frac{1}{2}\int_{t}^{t+T} [x^{T}Qx + u^{T}Pu]dt, \qquad (27)$$

where T > 0 is a constant prediction horizon, $S \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{r \times r}$ are as usual given symmetric weighting matrices.

From the maximum principle and $\dot{x} = \frac{\partial H}{\partial p}$, $\frac{\partial H}{\partial u} = 0$ and $\dot{p} = -\frac{\partial H}{\partial x}$ where H is the Hamiltonian function, we may deduce the autonomous system

$$\dot{\tilde{x}} = F\tilde{x} \tag{28}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ p \end{bmatrix}, \quad \dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix}.$$
(29)

Perform the following:

- 1. Find the system matrix F.
- 2. Use that p(t+T) = Sx(t+T) and find en expression for R in the linear relationship p = Rx.

Hints: You can use that the solution to the autonomous system (28) is given by

$$\tilde{x}(t) = e^{F(t-t_0)}\tilde{x}(t_0) = \Phi\tilde{x}(t_0)$$
(30)

where we also can partition the transition matrix $\Phi = e^{F(t-t_0)}$ as follows

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}.$$
(31)

We may express R as a function of the sub-matrices in Φ and S. Remarks : R is the solution to the Riccati equation of this problem.

Appendix

Continuous time optimal control

$$\dot{x} = f(x, u, t) \tag{32}$$

$$J = S(x(t_1)) + \int_{t_0}^{t_1} L(x, u, t) dt$$
(33)

The continuous time Maximum Principle

$$H = L + p^T f \tag{34}$$

$$\dot{p} = -\frac{\partial H}{\partial x} \tag{35}$$

$$p(t_1) = \frac{\partial S}{\partial x}|_{t_1} \tag{36}$$

Discrete time optimal control

$$x_{k+1} - x_k = f(x_k, u_k, k)$$
(37)

$$J_{i} = S(x_{N}) + \sum_{k=i}^{N-1} L(x_{k}, u_{k}, k)$$
(38)

The discrete time maximum Principle

$$H_k = L(x_k, u_k, k) + p_{k+1}^T(x_{k+1} - x_k)$$
(39)

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k} \tag{40}$$

$$p_N = \frac{\partial S}{\partial x_N} \tag{41}$$

Derivation rules

$$\frac{\partial}{\partial x}(Qx) = Q^T, \quad \frac{\partial}{\partial x}(x^TQx) = Qx + Q^Tx$$
(42)

$$\frac{\partial}{\partial x}((r-Dx)^TQ(r-Dx)) = -2D^TQ(r-Dx)$$
(43)