Discrete LQ optimal control with integral action

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1   Problem formulation

Given a proces model

$$x_{k+1} = Ax_k + Bu_k + v, \tag{1}$$
$$y_k = Dx_k + w, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^r$ is the control input vector, $y_k \in \mathbb{R}^m$ is the output (measurement) vector, and A, B and D are known system matrices of appropriate dimensions. The disturbances v and w are both unknown, i.e., v is an unknown constant or slowly varying process disturbance, and w are an unknown constant or slowly varying measurements noise vector.

We will study the LQ optimal controller subject to the following performance index,

$$J_i = \frac{1}{2}x_N^T S x_N + \frac{1}{2}\sum_{k=i}^{N-1}((y_k - r)^T Q(y_k - r) + \Delta u_k^T P \Delta u_k), \tag{3}$$

where $\Delta u_k = u_k - u_{k-1}$ and r is a reference signal and S, Q and P are symmetric positive semidefinite weighting matrices of appropriate dimensions. For large or infinite prediction horizon N or when S is chosen as the solution to Riccati equation of the problem, then we use the index

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}((y_k - r)^T Q(y_k - r) + \Delta u_k^T P \Delta u_k). \tag{4}$$

2   Problem solution

In order to solve the LQ optimal control problem we need a model which is independent of the unknown disturbances. For the sake of generality we are focusing on state space modeling.

From the state Equation (1) we have

$$\Delta x_{k+1} = A \Delta x_k + B \Delta u_k, \tag{5}$$

where $\Delta x_k = x_k - x_{k-1}$. From the measurement equation (2) we have

$$y_k = y_{k-1} + D\Delta x_k. \tag{6}$$

Augmenting (5) with (6) gives the state space model

$$\begin{bmatrix} \Delta x_{k+1} \\ y_k \end{bmatrix} = \begin{bmatrix} A & 0_{n\times m} \\ D & I_{m\times m} \end{bmatrix} \begin{bmatrix} \Delta x_k \\ y_{k-1} \end{bmatrix} + \begin{bmatrix} B \\ 0_{m\times r} \end{bmatrix} \Delta u_k, \tag{7}$$

$$y_k = \begin{bmatrix} D & I_{m\times m} \end{bmatrix} \begin{bmatrix} \Delta x_k \\ y_{k-1} \end{bmatrix}. \tag{8}$$

The performance index (3) with $r = 0$ and the state space model (7) and (8) are defining a standard LQ control problem. If r is a non-zero constant reference then the measurements equation (9) can be written as

$$y_k - r = y_{k-1} - r + D\Delta x_k. \tag{9}$$

The state and output equations (7) and (8) can then be rewritten as

$$\overbrace{\begin{bmatrix} \Delta x_{k+1} \\ y_k - r \end{bmatrix}}^{\tilde{x}_{k+1}} = \overbrace{\begin{bmatrix} A & 0_{n\times m} \\ D & I_{m\times m} \end{bmatrix}}^{\tilde{A}} \overbrace{\begin{bmatrix} \Delta x_k \\ y_{k-1} - r \end{bmatrix}}^{\tilde{x}_k} + \overbrace{\begin{bmatrix} B \\ 0_{m\times r} \end{bmatrix}}^{\tilde{B}} \Delta u_k, \tag{10}$$

$$\overbrace{y_k - r}^{\tilde{y}_k} = \overbrace{\begin{bmatrix} D & I_{m\times m} \end{bmatrix}}^{\tilde{D}} \overbrace{\begin{bmatrix} \Delta x_k \\ y_{k-1} - r \end{bmatrix}}^{\tilde{x}_k}. \tag{11}$$

Hence, we have a strictly proper state space model of the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \tag{12}$$
$$\tilde{y}_k = \tilde{D}\tilde{x}_k \tag{13}$$

The state space model (10) and (11) with the performance index (3) defines a standard LQ optimal control problem. The optimal control is of the form

$$\Delta u_k = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} \Delta x_k \\ y_{k-1} - r \end{bmatrix}, \tag{14}$$

which can be rewritten as

$$u_k = u_{k-1} + G_1\Delta x_k + G_2(y_{k-1} - r). \tag{15}$$

The LQ optimal controller (14) gives $y = r$ in steady state since the closed loop system is stable due to the properties of the LQ optimal controller. The states are seldom measured in practice. In this case we can use a state observer to define the deviation state Δx_k. However, another solution is to define Δx_k in terms of some past and known outputs $\dots, y_{k-1}, y_k$ and some known inputs $\dots, u_{k-1}$ and the model matrices A, B and D.

3 State observer for the state deviation

A common solution to the problem of estimating the state, x_k, in a model of the form (1) and (2) in which the noise is colored, is to include a random walk (integrator) in order to estimate the non-zero part of the noise, $\bar{v}_k$ in addition to the state estimate $\bar{x}_k$. This is necessary in order for the innovations to become white. One can thereafter form the state deviation $\Delta\bar{x}_k = \bar{x}_k - \bar{x}_{k-1}$, which is needed in the control algorithm. However, another more simple solution in this situation is to design an observer for the deviation model (5) and (6). This gives a state observer for $\Delta\bar{x}_k$ of the form

$$\Delta\bar{x}_{k+1} = A\Delta\bar{x}_k + B\Delta u_k + K(y_k - y_{k-1} - D\Delta\bar{x}_k), \tag{16}$$

where the initial estimate $\Delta\bar{x}_0$ should be specified. A natural choice is $\Delta\bar{x}_0 = 0$.

4 Connection with the PI controller

A conventional PI-controller can be written as

$$u = K_p\frac{1 + T_i s}{T_i s}(r - y) = K_p(r - y) + \frac{K_p}{T_i}\frac{1}{s}(r - y). \tag{17}$$

Defining the PI-controller state, z, as

$$z = \frac{1}{s}(r - y). \tag{18}$$

Hence, the PI controller can in continuous time be written as

$$\dot{z} = r - y, \tag{19}$$
$$u = K_p(r - y) + \frac{K_p}{T_i}z. \tag{20}$$

A discrete formulation of the PI controller is then

$$z_{k+1} - z_k = h(r - y_k), \tag{21}$$
$$u_k = K_p(r - y_k) + \frac{K_p}{T_i}z_k, \tag{22}$$

where h is the sampling interval. A deviation formulation of the PI controller is then found as follows

$$\begin{aligned} u_k - u_{k-1} &= K_p(r - y_k) + \frac{K_p}{T_i}z_k - (K_p(r - y_{k-1}) + \frac{K_p}{T_i}z_{k-1}) \\ &= -K_p(y_k - y_{k-1}) + \frac{K_p}{T_i}(z_k - z_{k-1}). \end{aligned} \tag{23}$$

From (21) we have that $z_k - z_{k-1} = h(r - y_{k-1})$. Substituting this into (23) gives

$$u_k = u_{k-1} + G_1(y_k - y_{k-1}) + G_2(y_{k-1} - r). \tag{24}$$

where

$$G_1 = -K_p, \ G_2 = -\frac{K_p}{T_i}h. \tag{25}$$

Furthermore, using that $y = Dx + w$ gives

$$u_k = u_{k-1} + G_1 D\Delta x_k + G_2(y_{k-1} - r). \tag{26}$$

The above discussion shows that the PI controller is exactly of the same structure as the LQ optimal controller (15). The difference is that the optimal controller takes feedback from the deviation state vector $\Delta x_k = x_k - x_{k-1}$ while the PI-controller only uses feedback from the output deviation $\Delta y_k = D\Delta x_k$.

5 Numerical examples

Example 5.1 (Isothermal chemical reactor)
A chemical isothermal reactor can be modeled as

$$\dot{x}_1 = \frac{u_1}{V}(u_2 - x_1) - skx_1^2, \tag{27}$$

$$\dot{x}_2 = -\frac{u_1}{V}x_2 + kx_1^2, \tag{28}$$

where $V = 1$, $k = 1$ and $s = 2$. The steady state control variables $u_1^s = 10$ and $u_2^s = 1$ gives the steady states $x_1^s = 0.8541$ and $x_2^s = 0.0729$. Linearizing around steady state gives the linear model

$$\Delta\dot{x} = A_c\Delta x + B_c\Delta u, \tag{29}$$

where $\Delta x = x - x^s$ and $\Delta u = u - u^s$ and

$$A_c = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x^s,u^s} = \begin{bmatrix} -\frac{u_1^s}{V} - 2skx_1^s & 0 \\ 2kx_1 & -\frac{u_1^s}{V} \end{bmatrix} = \begin{bmatrix} 13.4164 & 0 \\ 1.7082 & -10.0 \end{bmatrix}, \tag{30}$$

$$B_c = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}_{x^s,u^s} = \begin{bmatrix} \frac{u_2^s}{V} & \frac{u_1^s}{V} \\ -\frac{x_2^s}{V} & 0 \end{bmatrix} = \begin{bmatrix} 1.0 & 10.0 \\ -0.0729 & 0 \end{bmatrix}. \tag{31}$$

A discrete time model is obtained by using a zero order hold on the input and a sampling interval $h = 0.01$, i.e.,

$$x_{k+1} = Ax_k + Bu_k + v, \tag{32}$$

$$y_k = Dx_k, \tag{33}$$

where

$$A = e^{A_ch} = \begin{bmatrix} 0.8744 & 0 \\ 0.0152 & 0.9048 \end{bmatrix}, \tag{34}$$

4

$$B = A_c^{-1}(e^{A_c h} - I)B_c = \begin{bmatrix} 0.0094 & 0.0936 \\ -0.0006 & 0.0008 \end{bmatrix},$$
(35)

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v = x^s - Ax^s - Bu^s.$$
(36)

Choosing an LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{N}((y_k - r)^T Q(y_k - r) + \Delta u_k^T P \Delta u_k),$$
(37)

with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}, Q = \begin{bmatrix} 50 & 0 \\ 0 & 100 \end{bmatrix},$$
(38)

gives the LQ-optimal control

$$u_k = u_{k-1} + G_1 \Delta x_k + G_2(y_{k-1} - r),$$
(39)

where

$$G_1 = \begin{bmatrix} -15.7253 & 55.7233 \\ -1.9714 & -6.5884 \end{bmatrix}, G_2 = \begin{bmatrix} -4.7639 & 6.2149 \\ -0.3639 & -0.7540 \end{bmatrix}.$$
(40)

Simulation results after changes in the reference signal r are illustrated in Figure 1.

Example 5.2 (Van de Vusse chemical reactor)

A chemical isothermal reactor (Van de Vusse) is studied in this example. The relationship from the feed flow rate u into the reactor to the concentration of the product y at the outlet of the reactor is modeled by the following non-linear state space model.

$$\dot{x}_1 = -k_1 x_1 - k_3 x_1^2 + (v - x_1)u,$$
(41)
$$\dot{x}_2 = k_1 x_1 - k_2 x_2 - x_2 u,$$
(42)
$$y = x_2,$$
(43)

where the reaction rate coefficients are given by $k_1 = 50$, $k_2 = 100$, $k_3 = 10$. The concentration of the by-product into the reactor, v, is treated as an unknown constant or slowly varying disturbance with nominal value $v^s = 10$. Choosing a steady state control $u^s = 25$ gives the steady states $x_1^s = 2.5$ and $y^s = x_2^s = 1$.

A linearized model around steady state is given by

$$\Delta \dot{x} = A_c \Delta x + B_c \Delta u,$$
(44)

where $\Delta x = x - x^s$ and $\Delta u = u - u^s$ and

$$A_c = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x^s, u^s} = \begin{bmatrix} -k_1 - 2k_3 x_1^s - u^s & 0 \\ k_1 & -k_2 - u \end{bmatrix} = \begin{bmatrix} -125 & 0 \\ 50 & -125 \end{bmatrix},$$
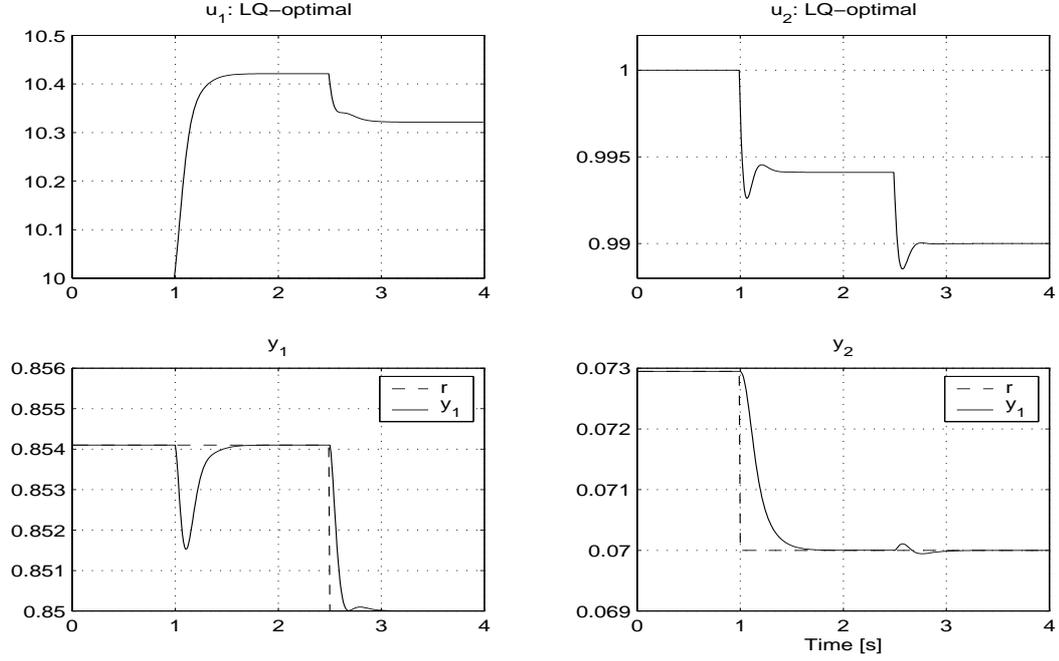(45)

Figure 1: Simulation of the chemical reactor in Example 5.1 with LQ-optimal control. This figure is generated by the MATLAB file **dlq_ex4_du.m**.

$$B_c = \left[\begin{array}{c} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{array} \right]_{x^s, u^s} = \left[\begin{array}{c} v^s - x_1^s \\ -x_2^s \end{array} \right] = \left[\begin{array}{c} 7.5 \\ -1 \end{array} \right]. \tag{46}$$

A discrete time model is obtained by using a zero order hold on the input and a sampling interval $h = 0.002$, i.e.,

$$x_{k+1} = Ax_k + Bu_k + v, \tag{47}$$
$$y_k = Dx_k, \tag{48}$$

where

$$A = e^{Ach} = \left[\begin{array}{cc} 0.7788 & 0 \\ 0.0779 & 0.7788 \end{array} \right], \tag{49}$$

$$B = A_c^{-1}(e^{Ach} - I)B_c = \left[\begin{array}{c} 0.0133 \\ -0.0011 \end{array} \right], \tag{50}$$

$$D = \left[\begin{array}{cc} 0 & 1 \end{array} \right], v = x^s - Ax^s - Bu^s + C(v - v^s). \tag{51}$$

Choosing an LQ criterion

$$J_i = \frac{1}{2} \sum_{k=i}^{\infty} (Q(y_k - r)^2 + P\Delta u_k^2), \tag{52}$$

6

with

$$P = 1, Q = 500,\qquad(53)$$

gives the LQ-optimal control

$$u_k = u_{k-1} + G_1 \Delta x_k + G_2(y_{k-1} - r),\qquad(54)$$

where

$$G_1 = \begin{bmatrix} -23.4261 & -84.5791 \end{bmatrix}, G2 = -20.0581.\qquad(55)$$

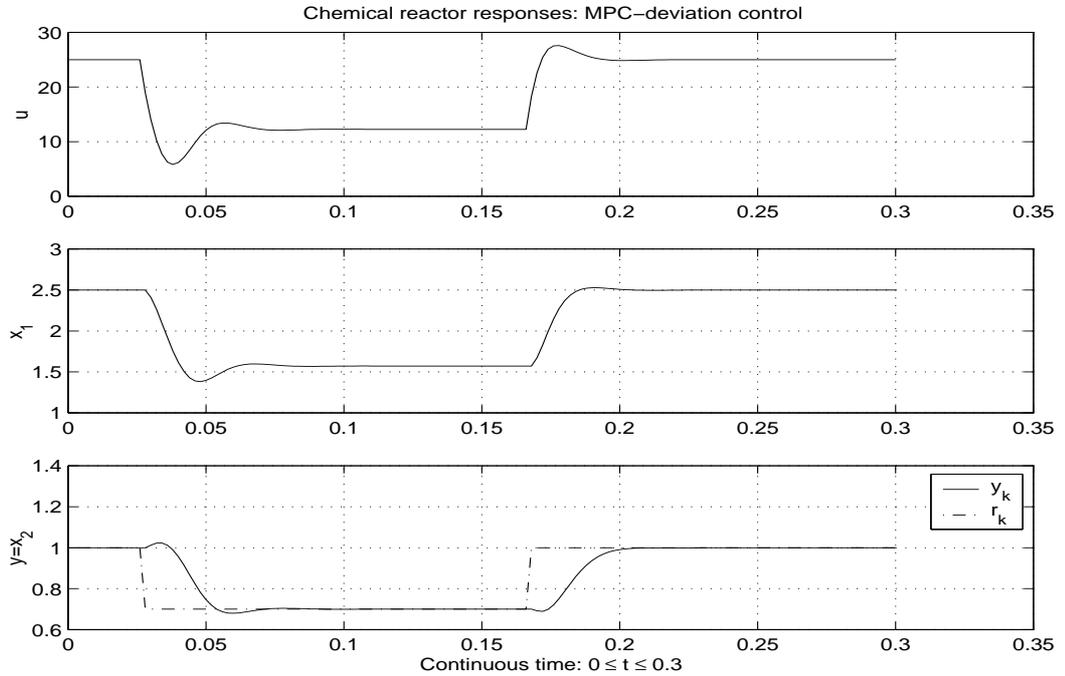*Simulation results after changes in the reference signal r are illustrated in Figure 2.*

Figure 2: Simulation of the chemical reactor in Example 5.2 with LQ-optimal control. This figure is generated by the MATLAB file **dlq_ex3_du.m**.

Example 5.3 (Distillation column)

One advantage of the presented LQ optimal control is that the control is designed in discrete time. Continuous processes with slow dominant dynamics are often controlled with a digital/discrete controller. If the sampling time is large then a continuous time controller design may give poor results when used as a discrete controller. We will here illustrate the simple discrete time LQ optimal control design for a distillation column.

Consider a distillation column with eight trays. Let the control variable $u_1 = R$ be the reflux to the column and $u_2 = V$ be the flow rate of vapor in the column. The composition of the top product $x_8 = x_D$ and the composition of the bottom product $x_1 = x_B$ are treated as measured output variables. The feed flow rate F and the composition x_F of the light product in F are both treated as unknown constant or slowly varying disturbances.

7

The continuous non-linear model with $n = 8$ states is first linearized around the steady state operating point $R^s = 2$, $V^s = 2.5$, $F^s = 1$ and $x_F^s = 0.5$. This gives a continuous time linear model of the form

$$\dot{\Delta x} = A_c \Delta x + B_c \Delta u + C_c \Delta v, \tag{56}$$
$$\Delta y = D \Delta x. \tag{57}$$

This model is then discretized with a sample interval of $h = 5$ [min]. This gives a discrete time model of the form

$$x_{k+1} = Ax_k + Bu_k + v_k, \tag{58}$$
$$y_k = Dx_k + w_k. \tag{59}$$

Choosing an LQ criterion

$$J_i = \frac{1}{2} \sum_{k=i}^{N} ((y_k - r)^T Q(y_k - r) + \Delta u_k^T P \Delta u_k), \tag{60}$$

with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q = 2500 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{61}$$

gives the LQ-optimal control

$$u_k = u_{k-1} + G_1 \Delta x_k + G_2(y_{k-1} - r), \tag{62}$$

where

$$G_1 = \begin{bmatrix} 12.8099 & 0.9303 & 0.3961 & -0.3187 & -1.5158 & -3.2992 & -6.3394 & -50.2082 \\ 30.3424 & 2.2003 & 1.4407 & 0.5590 & -0.6629 & -2.0994 & -4.1887 & -30.6734 \end{bmatrix}, \tag{63}$$

$$G_2 = \begin{bmatrix} 10.0833 & -29.5242 \\ 24.6664 & -17.1829 \end{bmatrix}. \tag{64}$$

The linear controller (62)-(64) on deviation form is in this example used to control the non-linear distillation column model with eight states. If the state vector is not available, then we may use a state observer or compute an expression for Δx_k from some past inputs and outputs, e.g. as in the Extende Model Predictive Control (EMPC) algorithm. Simulation results after changes in the reference signal r are illustrated in Figure 3.
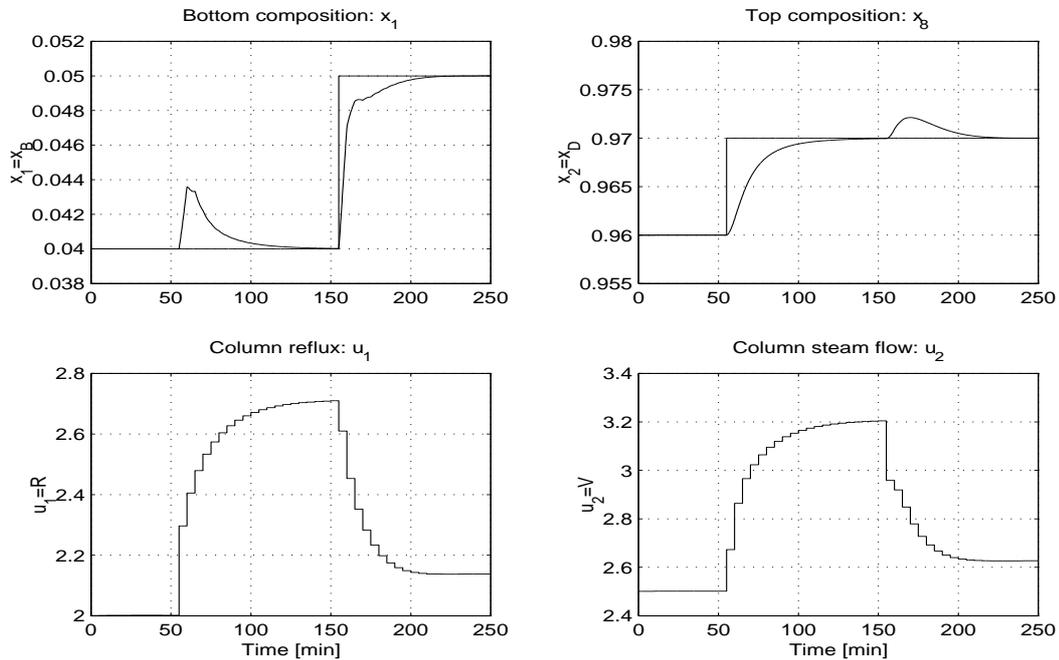
Figure 3: Simulation of the chemical reactor in Example 5.2 with LQ-optimal control. This figure is generated by the MATLAB file **main_kolreg_lq.m**.

5.1 MATLAB script for computation

```
function [G1,G2]=dlqdu_pi(A,B,D,Q,Rw);
% DLQDU_PI
% [G1,G2]=dlqdu_pi(A,B,D,Q,R);
% Purpose
% Compute LQ-optimal control feedback matrices G1 and G2 for
% the controller
% u=u+G1*(x-x_old)+G2*(y_old-r);
% On input
% A,B,D - discrete state space model matrices.
% Q - Wheigting matrix for the output vector y_k.
% R - Weighting matrix for the control deviation vector, Delta u_k=u_k-u_(k-1).

%% Make augmented state space model for LQ-design.
nx=size(A,1); nu=size(B,2); ny=size(D,1);
At=[A,zeros(nx,ny);D,eye(ny,ny)];Bt=[B;zeros(ny,nu)];Dt=[D,eye(ny,ny)];
Qt=Dt'*Q*Dt;

%% Solve Riccati-equation and compute feedback matrix.
[K,Rr]=dlqr(At,Bt,Qt,Rw);
G=-K;
G1=G(:,1:nx); G2=G(:,nx+1:nx+ny);
```

9